$z = \vec{10} \rightsquigarrow (dR_{\vec{10}})^{-1} \circ B_{\vec{10}} \rightsquigarrow$ MZV's

Φ_{KZ}

Rem) $\exists$ p-adic analogue

(*. cpx conj. invariant $\exists!$ Betti path $B_{\vec{10}}$
$\longmapsto$ Besser's Frob. inv. path.

(*. $\pi_1^{crys} \overset{\exists}{\cong} \mathbb{Q}_p \otimes_{\mathbb{Q}} \pi_1^{dR}$ Berthelot-Ogus type isom.
$\mathbb{Z}_p \to \mathbb{Q}_p \to \mathbb{F}_p$

no iterated integral

Frob. action $\rightsquigarrow$ iterated integral

§3-2 mixed Tate motives

k : field of char $= 0$

$DM_{gm}(k)$: Voevodsky's triangulated cat. of mixed motives

$MT(\mathbb{Z})$: the Tann. cat. of mixed Tate motives /$\mathbb{Z}$
rigorously def'd

localize $K^b(SmCor(k))$
by the thick subcat.

$\exists$ path
$\vec{1}$ to z on $U(\mathbb{C})$

$(U(\mathbb{C}); \vec{01}, z)$

$H^1(U, \mathcal{O}_U) = 0 \rightsquigarrow {}^{\forall}\pi_1^{dR}(U/\mathbb{Q}; \vec{01})$-torsor is trivial

$\rightsquigarrow \exists! \, dR_z \in \pi_1^{dR}(U/\mathbb{Q}; \vec{01}, z)$

Sm Co

K^b(

Th(

no iterated integral

§3-2 mixed Tate motives

k : field of char $= 0$

$DM_{gm}(k)$: Voevodsky's triangulated cat. of mixed motives

$\downarrow$

$MT(\mathbb{Z})$: the Tann. cat. of mixed Tate motives $/\mathbb{Z}$
$\uparrow$ rigorously def'd

$SmCor(k)$: Obj smooth k-schemes
Mor $Hom(X,Y)$
$\quad\| $
$\langle Z \subset X \times Y \to X \rangle_{\mathbb{Z}}$ ← generate
(red irred closed; finite surj over a conn. comp. of X)

$K^b(SmCor(k))$: the cat. of bdd complexes of $SmCor(k)$

$\exists$ path $\vec{1}$ to z on $U(\mathbb{C})$

$(U(\mathbb{C}); \vec{01}, z)$

$H^1(U, \mathcal{O}_U) = 0 \rightsquigarrow {}^\forall \pi_1^{dR}(U/\mathbb{Q}; \vec{01})$-torsor is trivial

$\rightsquigarrow \exists!\, dR_z \in \pi_1^{dR}(U/\mathbb{Q}; \vec{01}, z)$

Th (Chen)

$\mathbb{C} \otimes_{\mathbb{Q}} \pi_1^B(U(\mathbb{C}), \vec{01}, z) \cong \mathbb{C} \otimes_{\mathbb{Q}} \pi_1^{dR}(U/\mathbb{Q}, \vec{01}, z)$

$\underbrace{(dR_z)^{-1} \circ B_z}$ $\in \pi_1^{dR}(U/\mathbb{Q}, \vec{01})(\mathbb{C})$

$\hookrightarrow \mathbb{C}\langle\langle A, B \rangle\rangle$

$G_0(z) = \sum_w (-1)^{\text{depth}(w)} Li_w(z)$

dR_z, B_z, z, $\vec{01}$

- localize $K^b(\mathrm{SmCor}(k))$
  by the thick subcat.
  generated by

$$\begin{cases} [X \times \mathbb{A}^1] \to [X] & \text{homotopy inv.} \\ [U \cap V] \to [U] \oplus [V] \to [X] & \text{Mayer-Vietoris for } X = U \cup V \text{ open} \end{cases}$$

- add the image of idempotents
- invert $\mathbb{Z}(1)$ $(= \mathbb{G}_m[-1])$

$\to DM_{gm}(k)$

$DM_{gm}(k)_{\mathbb{Q}}$ ← tensoring Hom by $\mathbb{Q}$

$\cup$

$DMT(k)$: the full subcat. gen. by $\mathbb{Q}(i)$

$M \in \mathrm{Obj}(DMT(k))$

$DMT(k)^{\geq 0}$: $gr^W_{-2n} M \cong \bigoplus_i \mathbb{Q}(n)[m_i]$, $m_i \leq 0$ $\forall i$

$DMT(k)^{\leq 0}$: $m_i \geq 0$ $\forall i$

k : number field $\supset \mathcal{O}_S$: ring of S-

Deligne-Gonch

$\longrightarrow DM_{gm}(k)$

$DM_{gm}(k)_{\mathbb{Q}}$ ← tensoring Hom by $\mathbb{Q}$

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$DMT(k)$: the full sub cat. gen. by $\mathbb{Q}(1)$

$DMT(k)^{\geq 0}$ ← $M \in Obj(DMT(k))$, $gr^W_{-2n} M \cong \bigoplus_i \mathbb{Q}(n)[m_i]$, $m_i \leq 0$ $\forall i$

$DMT(k)^{\leq 0}$ ← $m_i \geq 0$ $\forall i$

If k holds Beilinson-Soulé vanishing conj (If k: number field, OK)

$\left(\begin{array}{l} K_{2n-i}(k)^{(n)}_{\mathbb{Q}} = 0 \text{ for } \forall i < 0 \\ \quad \parallel \\ H^i_M(k, \mathbb{Q}(n)) \end{array} \right)$

$\Rightarrow (DMT(k)^{\geq 0}, DMT(k)^{\leq 0})$ t-str. of $DMT(k)$

$\xrightarrow{[BBD]} MT(k) := DMT(k)^{\geq 0} \cap DMT(k)^{\leq 0}$

↑ abel. cat. $\otimes$ $\rightsquigarrow$ Tann. cat.

k : number field $\supset O_S$ ring of S-integers

(S : a fin. set of primes of k)

Deligne-Goncharov

$MT(k) \supset MT(O_S)$

$\uparrow$ appearing

by sub quot. $\mathrm{Ext}^1_{MT(k)}(\mathbb{Q}(0), \mathbb{Q}(1)) \cong k^\times \otimes \mathbb{Q}$

live in $O_S^\times \otimes \mathbb{Q}$

DG constructed $\pi_1^{M}(\mathbb{U}; a, b) \in \text{pro-}MT(\mathbb{Z})$

$a, b \in \{\vec{01}, \vec{10}, \vec{0\infty}, \vec{\infty 0}, \vec{1\infty}, \vec{\infty 1}\}$

motivic fundamental groupoid of $\mathbb{U}$

s.t. $B(\pi_1^M) = \pi_1^B$, $dR(\pi_1^M) = \pi_1^{dR}$

Betti real'n de Rham real'n

(cobar construction (Beilinson))

of S-integers

... set of primes of k

$| = k^\times \otimes \mathbb{Q}$

$\otimes \mathbb{Q}$

DG constructed $\pi_1^M(\mathbb{U}; a, b) \in$ pro-$MT(\mathbb{Z})$

$a, b \in \{\vec{01}, \vec{10}, \vec{0\infty}, \vec{\infty 0}, \vec{1\infty}, \vec{\infty 1}\}$

<u>motivic fundamental groupoid</u> of $\mathbb{U}$

s.t. $B(\pi_1^M) = \pi_1^B$, $dR(\pi_1^M) = \pi_1^{dR}$

Betti real'n, de Rham real'n

(cobar construction (Beilinson))

$\mathcal{T}$: Tann. cat.

$A \otimes A \xrightarrow{\mu} A \rightsquigarrow$ ring obj. of $\mathcal{T}$

$\rightsquigarrow$ affine scheme obj. of $\mathcal{T}$

$\rightsquigarrow$ affine gp scheme obj. of $\mathcal{T}$

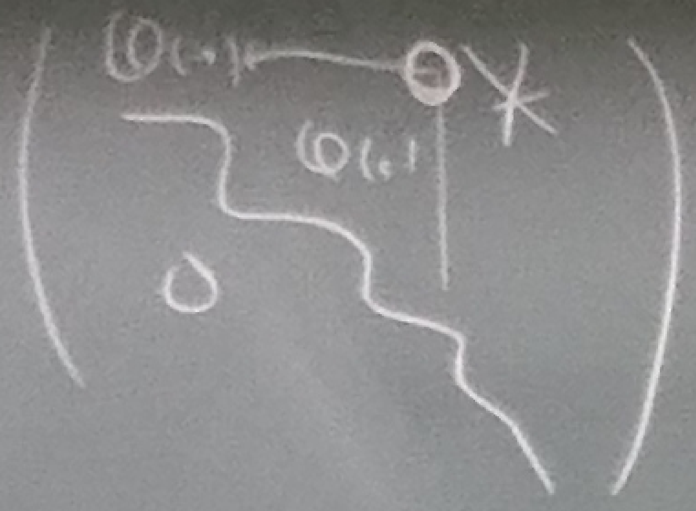
$\rightsquigarrow$ groupoid obj. of $\mathcal{T}$

$\omega : MT(\mathbb{Z}) \longrightarrow Vect_{\mathbb{Q}}$

can. fiber functor

$M \longmapsto \bigoplus_{n\in\mathbb{Z}} Hom(\mathbb{Q}(n), gr^W_{-2n} M)$

$(O_S = \mathbb{Z} \Rightarrow \omega = dR)$



$Gal(MT(\mathbb{Z}), dR) := Aut^{\otimes}(MT(\mathbb{Z}), dR)$

motivic Gal gp of $MT(\mathbb{Z})$

$= \mathbb{G}_m \ltimes U_{dR}$

pro-unip. gp

$Lie(U_{dR}^{ab}) = \prod_{n\geq 1} Ext^1_{MT(\mathbb{Z})}(\mathbb{Q}(0), \mathbb{Q}(n))^{\vee}$

$\cong \prod_{n\geq 1} K_{2n-1}(\mathbb{Z})_{\mathbb{Q}}^{\vee}$

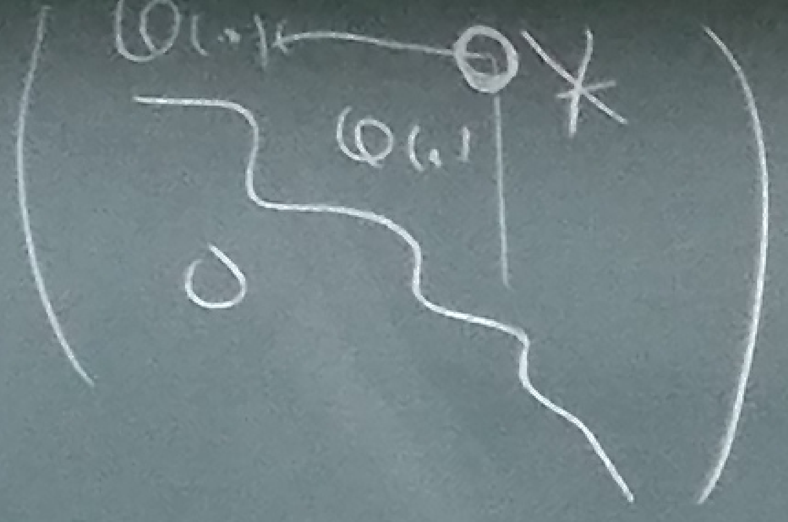
π_1^{dR} comes from π_1^{M}

$\rightsquigarrow Gal(MT(\mathbb{Z}), dR) \xrightarrow{\iota} Aut\, \pi_1^{dR(\mathbb{P}^1 \setminus \{0,1,\infty\})}$

Th (Brown ; motivic Belyi)

ι : injective

(cf.



$$\mathrm{Lie}(U_{dR}^{ab}) = \prod_{n\geq 1} \mathrm{Ext}^1_{MT(\mathbb{Z})}(\mathbb{Q}(0), \mathbb{Q}(n))^\vee$$
$$\cong \prod_{n\geq 1} K_{2n-1}(\mathbb{Z})_{\mathbb{Q}}^\vee$$

π_1^{dR} comes from π_1^{M}

$$\rightsquigarrow \quad \mathrm{Gal}(MT(\mathbb{Z}), dR) \xrightarrow{\ \iota\ } \mathrm{Aut}\, \pi_1^{dR}(U/\mathbb{Q}, \vec{01})$$

<u>Th</u> (Brown, motivic Belyi)

ι: injective

$\left(\text{cf. } \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \xhookrightarrow{\text{Belyi}} \mathrm{Aut}\, \pi_1(U, \vec{01})\right)$

$\mathbb{Z}), dR)$

$MT(\mathbb{Z})$

γ^p